Complex Analysis.

Mapping of a upper half Plane :

Statement :

The bi-linear transformation which maps $\text{Im } z \geq 0$ (upper half Plane) onto a unit is $|w| \leq 1$ is given by $W = e^{i\lambda}\left[\frac{z-\alpha}{z-\bar{\alpha}}\right]$ Provided $\text{Im } \alpha > 0$

Proof :

Any bi-linear transformation is of the form

$$W = \frac{az+b}{cz+d} \quad \text{①} \quad ad - bc \neq 0$$

WKT,

Bi-linear transformation transforms circle (or) straight line to circle (or) straight line and inverse point to inverse point.

Here the real axis $\text{Im } z = 0$ is transformed to unit circle $|W| = 1$.

As 0 and ∞ are inverse Point with respect to $|W| = 1$ their Pre-image $z = \frac{-b}{a}$ & $z = \frac{-d}{c}$ are symmetric Points.

Let $\alpha = \frac{-b}{a}$, $\bar{\alpha} = \frac{d}{c}$

Hence, ① takes the form

$$\therefore W = \frac{az+b}{cz+d} = \frac{a(z+b/a)}{c(z+d/c)}$$

$$W = \frac{a}{c}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right) \quad \text{②}$$

As $z = 0$ is Pre-image same Point on $|W| = 1$.

we have. $1 = |W| = \left|\frac{a}{c}\right| \left|\frac{0-\alpha}{0-\bar{\alpha}}\right|$

$$1 = \left|\frac{a}{c}\right|$$

$$1 = \frac{|a|}{|c|}$$

$$\frac{a}{c} = e^{i\lambda}$$

where λ is real thus the transformation takes the form $W = e^{i\lambda}\frac{(z-\alpha)}{(z-\bar{\alpha})}$

Now, $z=\alpha$ is Pre-image of $w=0$, since $w=0$ being centre of the circle $|W|=1$ the Pre-image z should be interior point of the domain.

(ie) $Im\,\alpha > 0$.

Thus bi-linear transformation which takes $Im\, z \geq 0$ on the unit disk given by

$$W = e^{i\lambda}\frac{(z-\alpha)}{(z-\bar{\alpha})} \quad \text{Provided } Im\,\alpha > 0.$$

verify:

To verify $W = e^{i\lambda}$ is the transformation which takes $Im\, z \geq 0$ onto $|W| = 1$.

$$W\bar{W} - 1 = \frac{e^{i\lambda}(z-\alpha)}{(z-\bar{\alpha})} = \frac{e^{i\lambda}\overline{(z-\alpha)}}{\overline{(z-\bar{\alpha})}} - 1$$

$$= \frac{(z-\alpha)(\bar{z}-\bar{\alpha})}{(z-\bar{\alpha})(\bar{z}-\alpha)} - 1$$

$$= \frac{(z\bar{z}-\alpha\bar{z}-\bar{\alpha}z+\alpha\bar{\alpha}) - (z-\bar{\alpha})(\bar{z}-\alpha)}{|z-\bar{\alpha}|^2}$$

$$= \frac{z\bar{z} - \alpha\bar{z} - \bar{\alpha}z + \alpha\bar{\alpha} - z\bar{z} + \bar{\alpha}\bar{z} + \alpha z - \alpha\bar{\alpha}}{|z-\bar{\alpha}|^2}$$

$$= \frac{z(\alpha - \bar{\alpha}) + \bar{z}(\bar{\alpha} - \alpha)}{|z-\bar{\alpha}|^2}$$

$$= \frac{(\alpha - \bar{\alpha})(z - \bar{z})}{|z-\bar{\alpha}|^2}$$

$$= \frac{(2i \, Im \, z) \cdot (2i \, Im \, \alpha)}{|z-\bar{\alpha}|^2}$$

$$= \frac{-4 (Im \, z)(Im \, \alpha)}{|z-\bar{\alpha}|^2}$$

$Im \, z = 0$ the real axis is transformed to

$W\bar{W} - 1 = 0$ ie) $|W| = 1$.

$W\bar{W} - 1 < 0$

$$\Rightarrow \frac{-4 (Im \, z)(Im \, \alpha)}{|z-\bar{\alpha}|^2} < 0$$

$\Rightarrow Im \, z \geq 0$.

ie) The upper half plane is mapped onto $|W| \leq 1$.

Hence $W = e^{i\lambda} \dfrac{(z-\alpha)}{(z-\bar{\alpha})}$

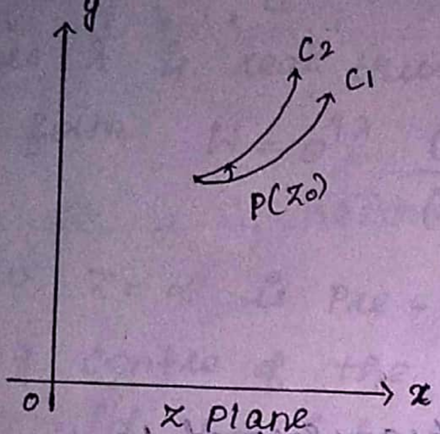
$Im \, \alpha > 0$ is transformation on which maps $Im \, z \geq 0$ onto the unit is $|W| < 1$.

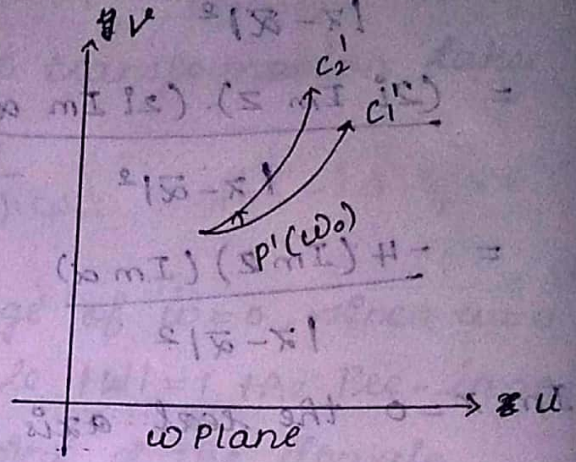
Conformal Mapping:

Eg: $W = e^{x}$

Suppose the transformation $u = v(x, y)$ and $v = u(x, y)$ maps the two curves c_1, c_2 intersecting at the point $P(z_0)$ of z-plane onto two curves

C_1', C_2' intersecting $P'(w_0)$ of w Plane. If the angle b/w C_1 and C_2 at z_0 is equal to the angle b/w C_1' and C_2' at w_0 both in magnitude and the sense of location the mapping is said to conformed.

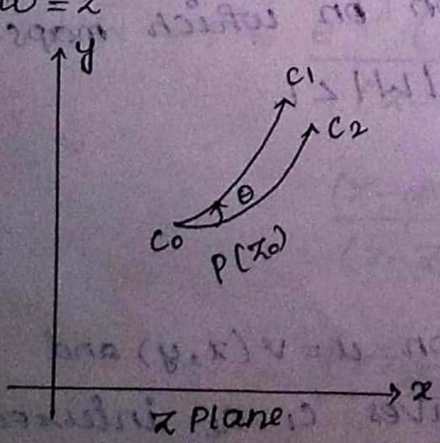


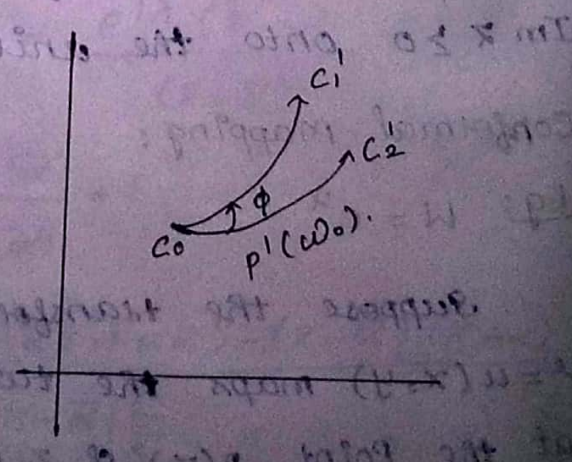


Isogonal mapping:

Suppose that the transformation $U = u(x,y)$ and $V = v(x,y)$ maps the two curve C_1, C_2 intersecting at the Point $P(z_0)$ of z-Plane onto the curves C_1', C_2' intersecting at $P'(w_0)$ of w Plane. If the angle b/w C_1 and C_2 at z_0 is equal to the angle b/w C_1' and C_2' at w_0 only in magnitude and opposite in sense, the mapping is said to be isogonal mapping.

Eg: $w = \bar{z}$





Sufficient condition for conformal Mapping:

Theorem:

If $w = f(z)$ is analytic and $f'(z) \neq 0$ in a domain D then the mapping $w = f(z)$ is conformal in D.

Proof:

Let $w = f(z)$ be an analytic function in a domain D in z-Plane. Let z_0 be the any interior points of D. Let c_1' and c_2' be the curve intersecting and at (w_0) of w-Plane.

If the angle b/w c_1 and c_2 at z_0 is equal to the angle b/w c_1' and c_2' at w_0 only in magnitude and opposite insense.

The corresponding two curves c_1 and c_2 intersecting at z_0 in z-Plane.

Let w_1 and w_2 be the point on the c_1' and c_2' respect to the corresponding points on z_1, z_2 on c_1, c_2 respectively such that the distance b/w z_1 and z_0 is equal to the distance b/w z_2 & z_0 (say r).

$\therefore$ We can write $z_1 - z_0 = r_1 e^{i\theta_1}$ and $z_2 - z_0 = r_2 e^{i\theta_2}$.

Let the tangent at z_0 the curves c_1 and c_2 make an angles α_1, α_2 with real axis.

so that $\theta_1 \to \alpha_1$, $\theta_2 \to \alpha_2$ as $r \to 0$.

Also, Let the tangent at w_0 to the curve c_1' & c_2' made angle β_1 and β_2 with real axes.

Let $\omega_1 - \omega_0 = \rho e^{i\phi_1}$ and $\omega_2 - \omega_0 = \rho_2 e^{i\phi_2}$

where $\phi_1 \to \beta_1$ as $\rho \to 0$ and $\phi_2 \to \beta_2$ as $\rho \to 0$.

$$f'(z_0) = \lim_{z_1 \to z_0} \frac{f(z_1) - f(z_0)}{z_1 - z_0}$$

$$= \lim_{z_1 \to z_0} \frac{\omega_1 - \omega_0}{z_1 - z_0} \Rightarrow \lim_{z_1 \to z_0} \frac{\omega_1 - \omega_0}{r_1 e^{i\theta_1}}$$

$$= \lim_{z_1 \to z_0} \frac{\rho_1 e^{i\phi_1}}{r_1 e^{i\theta_1}} = \lim_{z_1 \to z_0} \frac{\rho_1}{r_1} e^{i(\phi_1 - \theta_1)}$$

given $f'(z) \neq 0$.

$$f'(z_0) = Re^{i\lambda} \Rightarrow f'(z) = \lim_{z \to z_0} \frac{\rho_1}{r_1} e^{i(\theta_1 - \alpha_1)}$$

then

$$Re^{i\lambda} = \lim_{z_1 \to z_0} \frac{\rho_1}{r_1} e^{i(\beta_1 - \alpha_1)}$$

$$\Rightarrow R = \frac{\rho_1}{r_1} \text{ as } z_1 \to z_0 \text{ and}$$

$$\lambda = \lim_{z_1 \to z_0} \beta_1 - \alpha_1 \Rightarrow R_1 = \frac{\rho_1}{r_1}$$

$$\lambda = \beta_1 - \alpha_1 \quad \text{—— (1)}$$

III'ly

$$f'(z_0) = \lim_{z_2 \to z_0} \frac{f(z_2) - f(z_0)}{z_2 - z_0}$$

$$= \lim_{z_2 \to z_0} \frac{\rho_2 e^{i\phi_2}}{r_2 e^{i\theta_2}}$$

$$= \lim_{z_2 \to z_0} \frac{\rho_2}{r_2} e^{i(\beta_2 - \alpha_2)}$$

And we have,

$$\lambda = \lim_{z \to z_0} \beta_2 - \alpha_2$$

$$\lambda = \beta_2 - \alpha_2 \quad \text{—} \quad (2)$$

From (1) & (2)

$$\beta_1 - \alpha_1 = \beta_2 - \alpha_2$$

$$\beta_1 - \beta_2 = \alpha_1 - \alpha_2$$

ie) the angle b/w C_1' & C_2' at w_0 is equal in magnitude as well as in sign. to the angle b/w the curves C_1 and C_2 at z_0.

Thus the mapped $w = f(z)$ in conformal.

Hence Proved.

Unit - V

Eg: Find the linear transformation which maps $z_1 = -1$; $z_2 = 0$; $z_3 = 1$ onto the points $w_1 = -i$; $w_2 = 1$; $w_3 = i$

Proof :-

$$w = \frac{az+b}{cz+d} \quad ; \quad ad - bc \neq 0$$

Put $z = -1$

$$-i = \frac{a(-1)+b}{c(-1)+d}$$

$$\boxed{i(d-c) = b-a} \text{ ——— } 1$$

Put $z=0$

$$1 = \frac{b}{d} \Rightarrow \boxed{b=d} \text{ ——— } 2$$

Put $z=1$.

$$i = \frac{a+b}{c+d} \Rightarrow (a+b) = i\,(c+d)$$

$$\Rightarrow \boxed{(a+b) = i\,(b+c)} \text{ ——— } 3$$

From 2.

$$1 \Rightarrow -a+b = i\,(c-b) \text{ — } 4$$

$$3 + 4 \Rightarrow 2b = 2ic$$

$$\boxed{b = ci} \text{ — } 5$$

sub 5 in 3.

$$a + ic = i\,(ic + c)$$

$$a + ic = -c + ic$$

$$\boxed{a = -c}$$

$$w = \frac{-cz+ic}{cz+ic} = \frac{\cancel{c}(-z+i)}{\cancel{c}(z+i)}$$

$w = \dfrac{-z+i}{z+i}$ is the required transformation.

$$\frac{(w-1)+(w+1)}{(w-1)-(w+1)} = \frac{(z-2)(4-3i)+5z+10}{(z-2)(4-3i)-5z-10}$$

$$\frac{2w}{2} = \frac{-4z-8-3iz+6i+\cancel{5z}+10}{4z-8-3iz+6i-5z-10}$$

$$w = \frac{9z+2-3iz+6i}{-z-18-3iz+6i}$$

Lineal Transformation :

The transformation $w = Az+B$, $(A \neq 0)$ where A & B are complex constant, then it is said to be lineal transformation.

Magnification :

The mapping $w = Az$ when A is a non-zero complex constant and $z \neq 0$ is said to be magnification writing A and z is exponential form

$$A = ae^{i\alpha}, \quad z = re^{i\theta}$$

$$w = ae^{i\alpha} re^{i\theta}$$

$$= (ar)e^{i(\theta+\alpha)}$$

If $a>1$, the mapping expands if $a<1$, the mapping contracts the region in z-Plane.

if $a>0$, the rotaties is in +ve direction and if $a>0$ the rotaties in +ve the direction.

Then the mapping preserves the shape of the region.

Lineal Translation :

The translation $w = Az+B$ $(A \neq 0)$ where A & B are complex constant is said to be lineal translation.

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10m

The transformation $w = \frac{1}{z}$

The transformation $w = \frac{1}{z}$ is said to inversion discuss the transfolmation $w = \frac{1}{z}$ to show that inversion is combination of inversion with respect to unit circle followed by reflexion on real axis.

The mapping $w = \frac{1}{z}$ is one to one except at zero. consider the transformation.

$z = \frac{z}{(z)^2}$ this transformation is invension. with respect $|z| = 1$ two points P & Q are said to be inversion point with respect to any circle $|z| = r$ if.

(i) O, P, Q lies on a line.

(ii) P & Q lies on same side of O and

(iii) $\overline{OP} \cdot \overline{OQ} = r^2$

$$z = \frac{z}{|z|^2} = \frac{z}{z\bar{z}} = \frac{1}{\bar{z}}$$

by definition of $\frac{1}{z}$ is invension points of z w.r. to $|z| = 1$.

Now,

$$|z| = \frac{|z|}{|z|^2} = \frac{1}{|z|} \text{ and } \arg z = -\arg \bar{z} = \arg z$$

those points exterior to unit circle are mapped onto the non-zero points interior to it and vice versa. Any pt on the circle is mapped onto itself.

Now.

$w = \bar{z}$ $\left(w = \bar{z} = \left(\frac{\bar{z}}{z\bar{z}}\right) = \frac{\bar{z}}{\bar{z}.z} = \frac{1}{z}\right)$ is nothing

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but reflection of z on real axis. Extend $\pi(z) = \frac{1}{z}$ in the extended complex plane.

Linear fractional transformation : (or) Mobius Transformation

(E) The transformation of the form $w = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ when a, b, c, d are complex constant is known as linear fractional transformation.

moreover $w = \frac{az+b}{cz+d}$

$$w(cz+d) = az+b \quad \text{--- (1)}$$

$$wcz + dw = az+b$$

$$w(c-a)z + (dw-b) = 0 \quad \text{--- (2)}$$

From (1) & (2)

show that mobius transformation is linear both is w and z. Here it is also known as linear transformation.

Result :

Show that a linear transformation is combination of both elementary transformation.

WKT, $w = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ is a linear transformation.

$$= \frac{a(z+b/a)}{c(z+d/c)} = \frac{a}{c} \frac{(z+b/a+d/c-d/c)}{z+d/c}$$

$$= \frac{a}{c}\left[\frac{(z+d/c)+(b/a-d/c)}{(z+d/c)}\right]$$

$$= \frac{a}{c}\left[1 + \frac{bc-ad}{ac[z+d/c]}\right]$$

$$= \frac{a}{c} + \frac{bc-ad}{c^2[z+d/c]} \quad \text{--- (1)}$$

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Let $T_1 = z + d/c \rightarrow$ It is translation

$T_2 = \frac{1}{T_1} \rightarrow$ It is inversion.

$T_3 = \left[\frac{bc-ad}{c^2}\right] T_2{}^2 \rightarrow$ It is magnification

$w = \frac{a}{c} + T_3 \rightarrow$ It is translation.

Hence the bilinear translation is the combination of translation, inversion and magnification.

ie) It is combination of these elementary transformation.

Result : 2.

If $w = T(z) = \frac{az+b}{cz+d}$ is bilinear transformation then $z = T^{-1} w$ is also bilinear transformation.

sol:-

$$w = \frac{az+b}{cz+d}, \quad ad-bc \neq 0 \text{ is a BT.}$$

$$\Rightarrow czw + dw = az+b$$

$$(cw-a)z + (dw-b) = 0$$

$$\Rightarrow z = \frac{b-dw}{cw-a}$$

Now,

$z = T^{-1}(w) = \frac{b-dw}{cw-a}$ is B, T is $\begin{vmatrix} -d & b \\ c & -a \end{vmatrix} \neq 0$

$$\begin{vmatrix} -d & b \\ c & -a \end{vmatrix} = ad-bc \neq 0$$

$\therefore$ w is BT

Hence $z = T^{-1}(w)$ is also B.T.

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2. Prove that combination of B.T.s is also B.T

Proof :-

Let $T_1(z) = \dfrac{a_1 z + b_1}{c_1 z + d_1} = a_1 d_1 - b_1 c_1 \neq 0$

$T_2(z) = \dfrac{a_2 z + b_2}{c_2 z + d_2} = a_2 d_2 - b_2 c_2 \neq 0$

be also B.T's

$$T_1 \circ T_2(z) = T_1\left[T_2(z)\right]$$

$$= \frac{a_1\left[\dfrac{a_2 z + b_2}{c_2 z + d_2}\right] + b_1}{c_1\left[\dfrac{a_2 z + b_2}{c_2 z + d_2}\right] + d_1}$$

$$= \frac{(a_1 a_2 z + a_1 b_2 + b_1 c_2 z + b_1 d_2)/(c_2 z + d_2)}{(a_2 c_1 z + b_2 c_1 + d_1 c_2 z + d_1 d_2)/(c_2 z + d_2)}$$

$$= \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(a_2 c_1 + d_1 c_2) z + (b_2 c_1 z + d_1 d_2)} = \frac{Az + B}{Cz + D}$$

where, $A = a_1 a_2 + b_1 c_2$; $B = a_1 b_2 + b_1 d_2$

$C = a_2 c_1 + d_1 c_2$; $D = b_2 c_1 + d_1 d_2$.

(*) is Provided $AD - BC \neq 0$

Consider,

$$AD - BC = [a_1 a_2 + b_1 c_2][b_2 c_1 + d_1 d_2] - [a_1 b_2 + b_1 d_2][a_2 c_1 + d_1 c_2]$$

$$= a_1 a_2 \cancel{b_2 c_1} - a_1 a_2 \cancel{b_2} c_1 + a_1 a_2 d_1 d_2 + b_1 b_2 c_1 c_2 + b_1 c_1 d_1 d_2 - a_2 b_1 c_1 d_2 - b_1 c_2 d_1 d_2 + a_1 b_2 c_2 d_1$$

$$= a_1 d_1 (a_2 d_2 - b_2 c_2) - b_1 c_1 (a_2 d_2 - b_2 c_2)$$

$$= (a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2)$$

$\neq 0$.

$\therefore [a_1 d_1 - b_1 c_1 \neq 0 \ \& \ a_2 d_2 - b_2 c_2 \neq 0]$

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T.8 Hence $T_1 O T_2$ is also B.T.

Critical Points :

The Points $z = -b/a$ and $z = -d/c$ which Corresponds to $w=0$ and $w = +\infty$, the critical Points of B.T. similarly the points $w = b/d$ and $w = a/c$ which corresponds to $z=0$ and $z=\infty$ are also critical Points.

10m 5m Theorem :

P.T a B.T takes circles or straight lines to circles or straight line and inverse points to inverse points.

Proof:- WKT, the equation of the circle having inverse Points p & q can be written as.

$$\left|\frac{z-p}{z-q}\right| = k, \quad (k \neq 1) \quad \text{--- (1)}$$

if $k \neq 1$ then (1) represents a straight line.

let $w_2 = \frac{az+b}{cz+d}$, $ad-bc \neq 0$

$wcz + dw - az - b = 0$

$z(wc-a) = b - dw$

$$z = \frac{-dw+b}{cw-a} \quad \text{--- (2)}$$

sub (2) in (1)

$$\left|\frac{-dw-b-p}{cw-a-q}\right| = k$$

$$\Rightarrow \left|\frac{-dw+b-cpw+ap}{-cw+b-cqw+aq}\right| = k$$

$$\Rightarrow \left|\frac{-(cp+d)w+(ap+b)}{-(cq+d)w+(aq+b)}\right| = k$$

$$\Rightarrow \left| \frac{w - \frac{ap+b}{cp+d}}{w - \frac{aq+b}{cq+d}} \right| = \frac{cq+d}{cp+d} = k$$

$$\left| \frac{w-p'}{w-q'} \right| = k' \quad \text{—— (3)}$$

(3) respresents a circle in w-Plane where p' & q' are inverse points.

* if $k=1$ & $k'=1$ the B.T straight line to str. line and symmetric Points to symmetric Points.

* If $k=1$ & $k' \neq 1$ st. line is transformed to circle and symmetric Points mapped onto inverse Points.

* If $k \neq 1$, $k'=1$ Circle is mapped onto st. line & inverse Points are mapped onto symmetric Points.

* If $k \neq 1$, $k' \neq 1$ the BT transform to circle to Circle and inverse Points to inverse Points.

An Implicit Form:

Cross ratio:

Cross Ratio of four Points z_1, z_2, z_3, z_4 defined by $\dfrac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ and it is denoted as (z_1, z_2, z_3, z_4).

Theorem:

10m or 5m

* The B.T Preserves Cross Ratio [OR] Cross ratio is invariant under B.T

Proof :-

Let $w = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$ be B.T

$$w_i - w_j = \frac{az_i+b}{cz_i+d} - \frac{az_j+b}{cz_j+d}$$

$$= \frac{(acz_iz_j + bcz_j + adz_j + bd) - (acz_iz_j + adz_j + bcz_i + bd)}{(cz_i+d)(cz_j+d)}$$

$$= \frac{ad(z_i - z_j) - bc(z_i - z_j)}{(z_i+d)(z_j+d)} = \frac{(ad-bc)(z_i-z_j)}{(z_i+d)(z_j+d)}$$

using.

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(ad-bc)(z_1-z_2)(ad-bc)(z_3-z_4)}{(cz_1+d)(cz_2+d)(cz_3+d)(cz_4+d)} \times \frac{(cz_2+d)(cz_3+d)(cz_4+d)(cz_1+d)}{(ad-bc)(z_2-z_3)(ad-bc)(z_4-z_1)}$$

$$= \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

Thus $(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$ is Cross ratio invariant bilinear transformation.